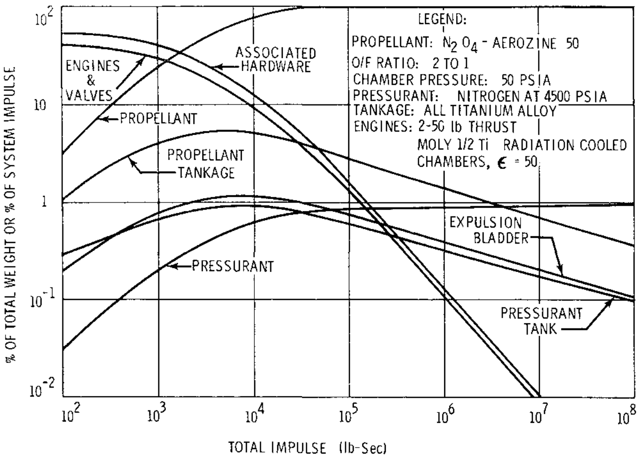

Fig. 2 Effect of total impulse on system component weights.

system. If the total amount of propellant is to be divided between "n" modules, then a weight penalty can be incurred. However, for medium to high total impulse ($I_t \leq$ 10,000 lb-sec, $F \leq$ 1000 lb), this penalty need not be significant, and for some cases it can be reduced by the lower weights associated with less propellant lines and manifolds. Depending upon the type of system, it may be possible to increase system reliability using an independent modular system. This penalty can be shown to be insignificant for high total impulse, low thrust systems ($I_i >$ 100,000 lb-sec, $F <$ 1,000 lb) since the propellant and pressurant specific impulse are independent of the number of modules (n).

Figure 1 shows a typical modular reaction control propulsion system, which was analyzed based upon the equations shown in Table 1. Figures 2 and 3 show the results of these calculations for various total impulses. In this case, the engines and associated hardware (values, lines, etc.) comprise a majority of the system weight for the low total impulse system ($I_t \leq$ 2000 lb-sec). As the total impulse increases, the majority of weight, as expected, is propellant. The pressurant approaches a maximum of 1% of the total weight as the total impulse is increased. If the subsystem specific impulse equations for the propellant and pressurant are compared, their ratio is found to be constant. The effect of total impulse on the over-all system specific impulse for the case analyzed is shown in Fig. 3. As the total impulse increases, the system impulse approaches asymptotically to the theoretical propellant specific impulse. The ratio of the system to propellant specific impulse is the propulsion system weight factor. Only for an ideal propulsion system does this factor equal unity; for any real system, this is always less than unity, but as the total impulse increases, this factor approaches asymptotically to unity.

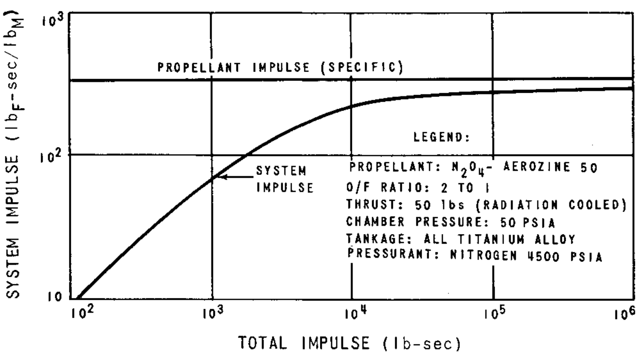

Fig. 3 Effect of total impulse upon system impulse.

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Application of Dynamic Programing to the Optimum Staging of Rockets

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Nomenclature†

$f_k[V]$ = minimum weight of rocket k achieving velocity V
g = gravitational const (32.174 ft/sec^2)
I_i = specific impulse of stage i
N_i = ratio of initial thrust to weight of rocket i
w_i = initial gross weight of stage i
$\sigma_i w_i$ = total jettison weight of stage i
W_i = initial gross weight of rocket $i \equiv W_{i+1} + w_i$
r_i = burnout mass ratio $= W_i/(W_{i+1} + \sigma_i w_i)$
v_i = velocity added during stage i
n = number of stages
W_L = payload weight $\equiv W_{n+1}$
V_{bo} = actual burnout velocity
V_t = design velocity $= V_{bo}$ plus losses

Introduction

THE use of the dynamic programing technique to optimize the staging of rockets was proposed by Ten Dyke[1] several years ago. Recently, Fan and Wan[2] applied a discrete version of the maximum principle to the same problem. They claim that their solution is computationally superior to that of Ten Dyke. My own admittedly limited experience with the use of dynamic programing would lead me to agree with them if they were considering a problem of the type usually treated by the calculus of variations. However, dynamic programing may sometimes be useful computationally for the

Table 1[a] Saturn C-5 vehicle

Stage, i	I_i, sec	Thrust, lb	Propellant weight, lb
1	300	7500×10^6	4400×10^6
2	400	1000	900
3	400	230	230

[a] W_L = 100,000 lb ond V_{bo} = 36,000 fps.

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† We will use Ten Dyke's original notation although it differs somewhat from that of Fan and Wan.

Table 2 Design configuration

i	W_i, lb	w_i, lb	r_i	N_i	v_i, fps
1	6,259,090	4,909,090	3.400	1.20	11,812
2	1,350,000	1,000,000	3.000	0.74	14,139
3	350,000	250,000	2.800	0.57	13,225
4	100,000	100,000	...	...	...

simpler class of problems, such as this one, which may be treated by the differential calculus (see Ref. 3, for example). In any case, the relative computational merits of two methods can only be established by examination of actual numerical results. To facilitate such a comparison, I wish to present a numerical example approximating one version of the Saturn C-5 configuration. (Of course, the ideal procedure would be to solve the same problem that was treated with the discrete maximum principle by the dynamic programing technique and then compare the results on the basis of accuracy, computation time, and computer storage requirements. However, since no numerical examples were presented in Ref. 2, only the dynamic programing side of the picture can be presented here.)

Dynamic Programing Formulation

If we assume rectilinear, field-free motion we can relate the velocity added to a rocket during a stage to the stage specific impulse and burnout mass ratio by

$$v_i = I_i g \ln r_i \tag{1}$$

The problem is to choose the r_i so that the payload is given a velocity

$$V_t = \sum_{i=1}^{n} v_i \tag{2}$$

in such a manner that the takeoff weight W_1 is a minimum. (The design velocity V_t is taken as the sum of the actual burnout velocity plus the losses due to gravity and the atmosphere.)

Using the definitions of W_i and r_i, and some simple manipulations, we can obtain an expression for the propellent weight:

$$(1 - \sigma_i)w_i = (r_i - 1)(W_{i+1} + w_i)/r_i \tag{3}$$

Next, we assume that the jettison weight is a function of the form

$$\sigma_i w_i = w_{xi} + \alpha_i w_i + \beta_i N_i W_i \tag{4}$$

where the w_{xi}, α_i, β_i are constants. Combining (3) and (4), using (1) and the definition of W_i, and rearranging, we obtain a recurrence relation for the W_i:

$$W_i = \frac{(1 - \alpha_i)W_{i+1} + w_{xi}}{\exp(-v_i/I_i g) - \alpha_i - \beta_i N_i} \tag{5}$$

[The derivation of the recurrence relation for a more general form of the jettison weight function than (4) is quite similar.]

Thus, the dynamic programing formulation requires the computation and tabulation of the following functions:

$$f_n(V) = \frac{(1 - \alpha_n)W_L + w_{xn}}{\exp(-v/I_n g) - \alpha_n - \beta_n N_n} \tag{6}$$

Table 3 Theoretical optimum

i	W_i, lb	w_i, lb	v_i, fps
1	5,464,611	2,494,230	5,106
2	2,970,380	2,428,032	17,124
3	542,348	442,348	17,045
4	100,000	100,000	...

Table 4 Takeoff weights, lb

Design configuration		6,259,090
Theoretical optimum	$\beta_i = 0$	5,464,611
Constrained optimum	$\beta_i = 0$	6,237,253
Theoretical optimum	$\beta_i \neq 0$	5,480,513
Constrained optimum	$\beta_i \neq 0$	6,252,257

$$f_i(V) = \min_{0 \le v_i \le V} \frac{(1 - \alpha_i)f_{i+1}(V - v_i) + w_{xi}}{\exp(-v_i/I_i g) - \alpha_i - \beta_i N_i} \tag{7}$$

$$i = n - 1, \ldots, 2, 1 \qquad 0 \le V \le Vt$$

[Equation (6) follows from applying (5) to the nth stage, and (7) follows from (5) and the principle of optimality.[4]]

The initial takeoff weight of the optimum rocket W_1 is $f_1(Vt)$. More detailed derivations of these equations may be found in Ref. 1 or 5.

Example

A reasonably accurate picture of the Saturn C-5 vehicle can be assembled from the unclassified literature. The mission we will consider is to put a 100,000 lb payload into an earth escape trajectory. We find the information in Table 1.‡

Assuming $\sigma_i = 0.1$, we find the results in Table 2. We will consider $W_1 = 6{,}259{,}090$ lb as the design takeoff weight, and use dynamic programing to investigate the optimality of the design.

The jettison weight parameters were first assumed to be $\alpha_i = 0.1$, $\beta_i = 0.0$, $w_{xi} = 0.0$; $i = 1, 2, 3$. The total velocity is the sum of the design v_i; thus $V_t = 39{,}176$ fps. The digital computer implementation of the dynamic programing formulation found the results in Table 3 to be optimum.

This is of course a purely theoretical result, since an important practical constraint, namely, the requirement that the upper two stages operate above the sensible atmosphere, will not be satisfied. We can make the results more realistic by incorporating this constraint. Also we can use more accurate values of the jettison weight parameters. (See Ref. 5 for details.) Some results are given in Table 4.

Computational Details

The results given in Table 3 required less than 2 min computing time on the IBM 7094. The total velocity of 39,176 fps was divided into 500 increments for the computations, and the results obtained are thus accurate to ±39 fps. The storage requirements are approximately

$$[850 + 11n + 5(b + 1) + 2(n - 1)(b + 1)]$$

decimal locations, where b is the number of velocity increments. In the example, $n = 3$, $b = 500$, and approximately 5400 locations were required.

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‡ See Refs. 6–8 for source of information in Table 1. Some of the numbers have been rounded.